# AUTOMORPHISM GROUPS OF AFFINE VARIETIES AND A CHARACTERIZATION OF AFFINE $n$-SPACE

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ABSTRACT. We show that the automorphism group of affine $n$-space $\mathbb{A}^n$ determines $\mathbb{A}^n$ up to isomorphism: If $X$ is a connected affine variety such that $\mathrm{Aut}(X) \simeq \mathrm{Aut}(\mathbb{A}^n)$ as ind-groups, then $X \simeq \mathbb{A}^n$ as varieties.

We also show that every finite group and every torus appears as $\mathrm{Aut}(X)$ for a suitable affine variety $X$, but that $\mathrm{Aut}(X)$ cannot be isomorphic to a semisimple group. In fact, if $\mathrm{Aut}(X)$ is finite dimensional and if $X \not\simeq \mathbb{A}^1$, then the connected component $\mathrm{Aut}(X)^\circ$ is a torus.

Concerning the structure of $\mathrm{Aut}(\mathbb{A}^n)$ we prove that any homomorphism $\mathrm{Aut}(\mathbb{A}^n) \to \mathcal{G}$ of ind-groups either factors through jac: $\mathrm{Aut}(\mathbb{A}^n) \to \mathbb{k}^*$, or it is a closed immersion. For $\mathrm{SAut}(\mathbb{A}^n) := \ker(\mathrm{jac}) \subseteq \mathrm{Aut}(\mathbb{A}^n)$ we show that every nontrivial homomorphism $\mathrm{SAut}(\mathbb{A}^n) \to \mathcal{G}$ is a closed immersion.

Finally, we prove that every non-trivial homomorphism $\varphi \colon \mathrm{SAut}(\mathbb{A}^n) \to \mathrm{SAut}(\mathbb{A}^n)$ is an automorphism, and that $\varphi$ is given by conjugation with an element from $\mathrm{Aut}(\mathbb{A}^n)$.

## 1. INTRODUCTION AND MAIN RESULTS

Our base field $\mathbb{k}$ is algebraically closed of characteristic zero. For an affine variety $X$ the automorphism group $\mathrm{Aut}(X)$ has the structure of an *ind-group*. We will shortly recall the basic definitions in the following section 2. The classical example is $\mathrm{Aut}(\mathbb{A}^n)$, the group of automorphism of affine $n$-space $\mathbb{A}^n = \mathbb{k}^n$.

The first main result shows that $\mathbb{A}^n$ is determined by its automorphism group.

**Theorem 1.1.** *Let $X$ be a connected affine variety. If $\mathrm{Aut}(X) \simeq \mathrm{Aut}(\mathbb{A}^n)$ as ind-groups, then $X \simeq \mathbb{A}^n$ as varieties.*

It is clear that $X$ has to be connected, since the automorphism group does not change if we form the disjoint union of $\mathbb{A}^n$ with a variety $Y$ with trivial automorphism group.

Another important question is which groups appear as automorphism groups of affine varieties. For finite groups we have the following result.

**Theorem 1.2.** *For every finite group $G$ there is a smooth affine curve $C$ such that $\mathrm{Aut}(C) \simeq G$.*

Moreover, there exist surfaces with infinite discrete automorphism groups (see [FK14, Proposition 7.5.2]). As for algebraic groups, we will see that every torus appears as $\mathrm{Aut}(X)$ (Example 6.4), but there are no examples where $\mathrm{Aut}(X) \simeq \mathrm{SL}_2(\mathbb{k})$. In fact, this is not possible as the next result shows.

**Theorem 1.3.** *Let $X$ be a connected affine variety. If $\dim \operatorname{Aut}(X) < \infty$, then either $X \simeq \mathbb{A}^1$ or $\operatorname{Aut}(X)^0$ is a torus.*

The last results concern the automorphism group $\operatorname{Aut}(\mathbb{A}^n)$ of affine $n$-space. This group has a closed normal subgroup $\operatorname{SAut}(\mathbb{A}^n)$ consisting of those automorphism $\mathbf{f} = (f_1, \ldots, f_n)$ whose Jacobian determinant $\operatorname{jac}(\mathbf{f}) := \det \left( \frac{\partial f_i}{\partial x_j} \right)_{(i,j)}$ is equal to 1:

$$\operatorname{SAut}(\mathbb{A}^n) := \ker(\operatorname{jac} \colon \operatorname{Aut}(\mathbb{A}^n) \to \Bbbk^*).$$

One could expect that $\operatorname{SAut}(\mathbb{A}^n)$ is simple as an ind-group, because its Lie algebra is simple, and that $\operatorname{SAut}(\mathbb{A}^n)$ is the only closed proper normal subgroup of $\operatorname{Aut}(\mathbb{A}^n)$. This is claimed in [Sha66, Sha81], but the proofs turned out to be not correct (see [FK14, Section 10]). What we can prove here are the following results.

**Theorem 1.4.**

(1) *Let $\varphi \colon \operatorname{Aut}(\mathbb{A}^n) \to \mathcal{G}$ be a homomorphism of ind-groups. Then either $\varphi$ factors through $\operatorname{jac} \colon \operatorname{Aut}(\mathbb{A}^n) \to \Bbbk^*$, or $\varphi$ is a closed immersion. This means that $\varphi(\operatorname{Aut}(\mathbb{A}^n)) \subseteq \mathcal{G}$ is a closed ind-subgroup and $\varphi \colon \operatorname{Aut}(\mathbb{A}^n) \xrightarrow{\sim} \varphi(\operatorname{Aut}(\mathbb{A}^n))$ is an isomorphism.*

(2) *Every nontrivial homomorphism $\operatorname{SAut}(\mathbb{A}^n) \to \mathcal{G}$ of ind-groups is a closed immersion.*

**Theorem 1.5.**

(1) *Every injective homomorphism $\varphi \colon \operatorname{Aut}(\mathbb{A}^n) \to \operatorname{Aut}(\mathbb{A}^n)$ is an isomorphism, and $\varphi = \operatorname{Int} \mathbf{g}$ for a well-defined $\mathbf{g} \in \operatorname{Aut}(\mathbb{A}^n)$.*

(2) *Every nontrivial homomorphism $\varphi \colon \operatorname{SAut}(\mathbb{A}^n) \to \operatorname{SAut}(\mathbb{A}^n)$ is an isomorphism, and $\varphi = \operatorname{Int} \mathbf{g}$ for a well-defined $\mathbf{g} \in \operatorname{Aut}(\mathbb{A}^n)$.*

Let us point out the following example from [FK14] showing that bijective homomorphisms of ind-groups are not necessarily isomorphisms. Denote by $\Bbbk\langle x, y \rangle$ the free associative $\Bbbk$-algebra in two generators. Then $\operatorname{Aut}(\Bbbk\langle x, y \rangle)$ is an ind-group, and we have a canonical homomorphism $\pi \colon \operatorname{Aut}(\Bbbk\langle x, y \rangle) \to \operatorname{Aut}(\Bbbk[x, y])$.

**Proposition 1.6** (FURTER-KRAFT [FK14])**.** *The map $\pi \colon \operatorname{Aut}(\Bbbk\langle x, y \rangle) \to \operatorname{Aut}(\Bbbk[x, y])$ is a bijective homomorphism of ind-groups, but it is not an isomorphism, because it is not an isomorphism on the Lie algebras.*

## 2. Notation and preliminary results

The notion of an ind-group goes back to Shafarevich who called these objects *infinite dimensional groups*, see [Sha66, Sha81]). We refer to [FK14] and [Kum02] for basic notation in this context and to [KZ14] and [KR15] for some important results.

**Definition 2.1.** An *ind-variety* $\mathcal{V}$ is a set together with an ascending filtration $\mathcal{V}_0 \subseteq \mathcal{V}_1 \subseteq \mathcal{V}_2 \subseteq \cdots \subseteq \mathcal{V}$ such that the following holds:

(1) $\mathcal{V} = \bigcup_{k \in \mathbb{N}} \mathcal{V}_k$;
(2) Each $\mathcal{V}_k$ has the structure of an algebraic variety;
(3) For all $k \in \mathbb{N}$ the subset $\mathcal{V}_k \subseteq \mathcal{V}_{k+1}$ is closed in the Zariski-topology.

A *morphism* between ind-varieties $\mathcal{V} = \bigcup_k \mathcal{V}_k$ and $\mathcal{W} = \bigcup_m \mathcal{W}_m$ is a map $\varphi \colon \mathcal{V} \to \mathcal{W}$ such that for every $k$ there is an $m$ such that $\varphi(\mathcal{V}_k) \subseteq \mathcal{W}_m$ and that the induced map $\mathcal{V}_k \to \mathcal{W}_m$ is a morphism of varieties. *Isomorphisms* of ind-varieties are defined in the usual way.

Two filtrations $\mathcal{V} = \bigcup_{k \in \mathbb{N}} \mathcal{V}_k$ and $\mathcal{V} = \bigcup_{k \in \mathbb{N}} \mathcal{V}'_k$ are called *equivalent* if for any $k$ there is an $m$ such that $\mathcal{V}_k \subseteq \mathcal{V}'_m$ is a closed subvariety as well as $\mathcal{V}'_k \subseteq \mathcal{V}_m$. Equivalently, the identity map $\mathrm{id} \colon \mathcal{V} = \bigcup_{k \in \mathbb{N}} \mathcal{V}_k \to \mathcal{V} = \bigcup_{k \in \mathbb{N}} \mathcal{V}'_k$ is an isomorphism of ind-varieties.

An ind-variety $\mathcal{V}$ has a natural topology where $S \subseteq \mathcal{V}$ is open, resp. closed, if $S_k := S \cap \mathcal{V}_k \subseteq \mathcal{V}_k$ is open, resp. closed, for all $k$. Obviously, a locally closed subset $S \subseteq \mathcal{V}$ has a natural structure of an ind-variety. It is called an *ind-subvariety*. An ind-variety $\mathcal{V}$ is called *affine* if all $\mathcal{V}_k$ are affine. A subset $X \subseteq \mathcal{V}$ is called *algebraic* if it is locally closed and contained in some $\mathcal{V}_k$. Such an $X$ has a natural structure of an algebraic variety.

**Example 2.2.** Any $\mathbb{k}$-vector space $V$ of countable dimension carries the structure of an (affine) ind-variety by choosing an increasing sequence of finite dimensional subspaces $V_k$ such that $V = \bigcup_k V_k$. Clearly, all these filtrations are equivalent.

If $R$ is a commutative $\mathbb{k}$-algebra of countable dimension, $\mathfrak{a} \subseteq R$ a subspace, e.g. an ideal, and $S \subseteq \mathbb{k}[x_1, \ldots, x_n]$ a set of polynomials, then the subset

$$\{(a_1, \ldots, a_n) \in R^n \mid f(a_1, \ldots, a_n) \in \mathfrak{a} \text{ for all } f \in S\} \subseteq R^n$$

is a closed ind-subvariety of $R^n$.

For any ind-variety $\mathcal{V} = \bigcup_{k \in \mathbb{N}} \mathcal{V}_k$ we can define the tangent space in $x \in \mathcal{V}$ in the obvious way. We have $x \in \mathcal{V}_k$ for $k \geq k_0$, and $T_x \mathcal{V}_k \subseteq T_x \mathcal{V}_{k+1}$ for $k \geq k_0$, and then define

$$T_x \mathcal{V} := \varinjlim_{k \geq k_0} T_x \mathcal{V}_k$$

which is a vector space of countable dimension. A morphism $\varphi \colon \mathcal{V} \to \mathcal{W}$ induces linear maps $d\varphi_x \colon T_x \mathcal{V} \to T_{\varphi(x)} \mathcal{W}$ for every $x \in X$. Clearly, for a $\mathbb{k}$-vector space $V$ of countable dimension and a for any $v \in V$ we have $T_v V = V$ in a canonical way.

The *product* of two ind-varieties is defined in the obvious way. This allows to define an *ind-group* as an ind-variety $\mathcal{G}$ with a group structure such that multiplication $\mathcal{G} \times \mathcal{G} \to \mathcal{G} \colon (g, h) \mapsto g \cdot h$, and inverse $\mathcal{G} \to \mathcal{G} \colon g \mapsto g^{-1}$, are both morphisms. It is clear that a closed subgroup $G$ of an ind-group $\mathcal{G}$ is an algebraic group if and only if $G$ is an algebraic subset of $\mathcal{G}$.

If $\mathcal{G}$ is an affine ind-group, then $T_e \mathcal{G}$ has a natural structure of a Lie algebra which will be denoted by $\mathrm{Lie}\,\mathcal{G}$. The structure is obtained by showing that every $A \in T_e \mathcal{G}$ defines a unique left-invariant vector field $\delta_A$ on $\mathcal{G}$, see [Kum02, Proposition 4.2.2, p. 114].

*Remark* 2.3. It is known that for $n \geq 2$ the Lie algebra $\mathrm{Lie}\,\mathrm{SAut}(\mathbb{A}^n)$ is simple and that $\mathrm{Lie}\,\mathrm{SAut}(\mathbb{A}^n) \subseteq \mathrm{Lie}\,\mathrm{Aut}(\mathbb{A}^n)$ is the only proper ideal, see [Sha81, Lemma 3]. Moreover, both Lie algebras are generated by the subalgebras $\mathrm{Lie}\,G$ where $G$ is an algebraic subgroup.

**Definition 2.4.** An ind-group $\mathcal{G} = \bigcup_k \mathcal{G}_k$ is called *discrete* if $\mathcal{G}_k$ is finite for all $k$. Clearly, $\mathcal{G}$ is discrete if and only if $\mathrm{Lie}\,\mathcal{G}$ is trivial.

An ind-group $\mathcal{G}$ is called *connected* if for every $g \in \mathcal{G}$ there is an irreducible curve $D$ and a morphism $D \to \mathcal{G}$ whose image contains $e$ and $g$.

The next result follows from [FK14, Theorem 3.1.1] and [KZ14, Theorem 4.3.2]. Here $\mathrm{Vec}(X)$ denotes the Lie algebra of (algebraic) vector fields on $X$, i.e. $\mathrm{Vec}(X) = \mathrm{Der}(\mathcal{O}(X))$, the Lie algebra of derivations of $\mathcal{O}(X)$.

**Proposition 2.5.** *Let $X$ be an affine variety. Then $\mathrm{Aut}(X)$ has a natural structure of an affine ind-group, and there is a canonical embedding $\xi\colon \mathrm{Lie}\,\mathrm{Aut}(X) \hookrightarrow \mathrm{Vec}(X)$ of Lie algebras.*

*Remark* 2.6. In case $X = \mathbb{A}^n$ the embedding $\xi$ identifies $\mathrm{Lie}\,\mathrm{Aut}(\mathbb{A}^n))$ with $\mathrm{Vec}^c(\mathbb{A}^n)$, the vector fields $\delta = \sum_i f_i \frac{\partial}{\partial x_i}$ with constant divergence $\mathrm{div}\,\delta = \sum_i \frac{\partial f_i}{\partial x_i} \in \mathbb{k}$, see [FK14, Proposition 3.5.1].

Another result which we will need is proved in [KZ14, Proposition 6.5.2].

**Proposition 2.7.** *Let $\varphi, \psi\colon \mathcal{G} \to \mathcal{H}$ be two homomorphisms of ind-groups. Assume that $\mathcal{G}$ is connected and that $d\varphi_e = d\psi_e\colon \mathrm{Lie}\,\mathcal{G} \to \mathrm{Lie}\,\mathcal{H}$. Then $\varphi = \psi$.*

A final result which we will use is the following (see [KRZ14, Lemma 6.1]). Denote by $\mathrm{Aff}_n \subseteq \mathrm{Aut}(\mathbb{A}^n)$ *the subgroup of affine transformations*, i.e. $\mathrm{Aff}_n = \mathrm{GL}_n(\mathbb{k}) \ltimes (\mathbb{k}^n)^+$. Similarly, the subgroup $\mathrm{SAff}_n \subseteq \mathrm{Aff}_n$ consists of the transformations with determinant 1, i.e. $\mathrm{SAff}_n = \mathrm{SL}_n(\mathbb{k}) \ltimes (\mathbb{k}^n)^+$.

**Proposition 2.8.** *Let $X$ be an affine variety with a faithful action of $\mathrm{SAff}_n$. If $\dim X \leq n$, then $X$ is $\mathrm{SAff}_n$-isomorphic to $\mathbb{A}^n$.*

*Remark* 2.9. It is shown in [KRZ14, Lemma 6.1] that the same holds if we replace $\mathrm{SAff}_n$ by $\mathrm{Aff}_n$. Using Theorem 1.5 we see that we can replace $\mathrm{SAff}_n$ by $\mathrm{Aut}(\mathbb{A}^n)$ or $\mathrm{SAut}(\mathbb{A}^n)$ as well.

## 3. The adjoint representation

If $L$ is a finitely generated Lie algebra, then $\mathrm{Aut}_{\mathrm{Lie}}(L)$ has a natural structure of an ind-group defined in the following way (see [FK14]). Choose a finite-dimensional subspace $L_0 \subseteq L$ which generates $L$ as a Lie algebra. Then the restriction map $\mathrm{End}_{\mathrm{Lie}}(L) \to \mathrm{Hom}(L_0, L)$ is injective and the image is a closed affine ind-subvariety. (To see this write $L$ as the quotient of the free Lie algebra $F(L_0)$ over $L_0$ modulo an ideal $I$.) Choosing a filtration $L = \bigcup_{k \geq 0} L_k$ by finite-dimensional subspaces, we set $\mathrm{End}_{\mathrm{Lie}}(L)_k := \{\alpha \in \mathrm{End}_{\mathrm{Lie}}(L) \mid \alpha(L_0) \subseteq L_k\}$ which is a closed subvariety of $\mathrm{Hom}(L_0, L_k)$ (see Example 2.2). Then we define the ind-structure on $\mathrm{Aut}_{\mathrm{Lie}}(L)$ by identifying $\mathrm{Aut}_{\mathrm{Lie}}(L)$ with the closed subset

$$\{(\alpha, \beta) \in \mathrm{End}_{\mathrm{Lie}}(L) \times \mathrm{End}_{\mathrm{Lie}}(L) \mid \alpha \circ \beta = \beta \circ \alpha = \mathrm{id}_L\} \subseteq \mathrm{End}_{\mathrm{Lie}}(L) \times \mathrm{End}_{\mathrm{Lie}}(L),$$

i.e.

$$\mathrm{Aut}_{\mathrm{Lie}}(L)_k := \{\alpha \in \mathrm{Aut}_{\mathrm{Lie}}(L) \mid \alpha, \alpha^{-1} \in \mathrm{End}_{\mathrm{Lie}}(L)_k\}.$$

It follows that $\mathrm{Aut}_{\mathrm{Lie}}(L)$ is an affine ind-group with the usual functorial properties.

**Lemma 3.1.** *Let $\rho\colon \mathcal{G} \to \mathrm{Aut}_{\mathrm{Lie}}(L)$ be an abstract homomorphism where $L$ is a finitely generated Lie algebra. Then $\rho$ is a homomorphism of ind-groups if and only if $\rho$ is an ind-representation, i.e. the map $\rho\colon \mathcal{G} \times L \to L$ is a morphism of ind-varieties.*

*Proof.* Assume that $L$ is generated by the finite dimensional subspace $L_0 \subseteq L$. If $\mathcal{G} = \bigcup_j \mathcal{G}_j$ and if $\rho \colon \mathcal{G} \times L \to L$ is a morphism, then, for any $j$, there is a $k = k(j)$ such that $\rho(\mathcal{G}_j \times L_0) \subseteq L_k$ and $\rho(\mathcal{G}_j^{-1} \times L_0) \subseteq L_k$. Hence, $\rho(\mathcal{G}_j) \subseteq \mathrm{Aut}_{\mathrm{Lie}}(L)_k$, and the map $\mathcal{G}_j \to \mathrm{Hom}(L_0, L_k)$ is clearly a morphism.

Now assume that $\mathcal{G} \to \mathrm{Aut}_{\mathrm{Lie}}(L)$ is a homomorphism of ind-groups. Then, for any $j$, there is a $k = k(j)$ such that $\rho(\mathcal{G}_j) \subseteq \mathrm{Aut}_{\mathrm{Lie}}(L)_k \hookrightarrow \mathrm{Hom}(L_0, L_k)$. Hence, $\rho(\mathcal{G}_j \times L_0) \subseteq L_k$, and $\mathcal{G}_j \times L_0 \to L_k$ is a morphism. $\qquad\square$

The *adjoint representation* $\mathrm{Ad} \colon \mathcal{G} \to \mathrm{Aut}_{\mathrm{Lie}}(\mathrm{Lie}\,\mathcal{G})$ of an ind-group $\mathcal{G}$ is defined in the usual way: $\mathrm{Ad}\,g := (d\,\mathrm{Int}\,g)_e \colon \mathrm{Lie}\,\mathcal{G} \xrightarrow{\sim} \mathrm{Lie}\,\mathcal{G}$ where $\mathrm{Int}\,g$ is the inner automorphism $h \mapsto ghg^{-1}$.

**Proposition 3.2.** *For any ind-group $\mathcal{G}$ the canonical map $\mathrm{Ad} \colon \mathcal{G} \to \mathrm{Aut}_{\mathrm{Lie}}(\mathrm{Lie}\,\mathcal{G})$ is a homomorphism of ind-groups.*

*Proof.* Let $\gamma \colon \mathcal{G} \times \mathcal{G} \to \mathcal{G}$ denote the morphism $(g, h) \mapsto ghg^{-1}$. For any $g \in \mathcal{G}$, the map $\gamma_g \colon \mathcal{G} \to \mathcal{G}$, $h \mapsto ghg^{-1}$, is an isomorphism of ind-groups, and its differential $\mathrm{Ad}(g) = (d\gamma_g)_e \colon \mathrm{Lie}\,\mathcal{G} \to \mathrm{Lie}\,\mathcal{G}$ is an isomorphism of Lie algebras. If $\mathcal{G} = \bigcup_k \mathcal{G}_k$, then for any $p, q \in \mathbb{N}$ there is an $m \in \mathbb{N}$ such that $\gamma \colon \mathcal{G}_p \times \mathcal{G}_p \to \mathcal{G}_m$. Clearly, $\mathrm{Ad}\,g$ for $g \in \mathcal{G}_p$ is then given by $(d\gamma_g)_e \colon T_e\mathcal{G}_q \to T_e\mathcal{G}_m$, and the map $\mathcal{G}_k \to \mathrm{Hom}(T_e\mathcal{G}_q, \mathcal{G}_m)$ is a morphism, by the following lemma. Now the claim follows from Lemma 3.1. $\quad\square$

**Lemma 3.3.** *Let $\Phi \colon X \times Y \to Z$ be a morphism of affine varieties and set $\Phi_x(y) := \Phi(x, y)$. Assume that there exist $y_0 \in Y$ and $z_0 \in Z$ such that $\Phi_x(y_0) = z_0$ for all $x \in X$. Then the induced map $X \to \mathrm{Hom}(T_{y_0}Y, T_{z_0}Z)$, $x \mapsto d_{y_0}\Phi_x$, is a morphism.*

*Proof.* We can assume that $Y, Z$ are vector spaces, $Y = W$ and $Z = V$. Choose bases $(w_1, \ldots, w_m)$ and $(v_1, \ldots, v_n)$. Then $\Phi$ is given by an element of the form

$$\sum_{i=1}^{n} \sum_{j} f_{ij} \otimes h_{ij} \otimes v_i, \text{ where } f_{ij} \in \mathcal{O}(X) \text{ and } h_{ij} \in \mathcal{O}(Y) = \Bbbk[y_1, \ldots, y_m],$$

and so the differential $(d\Phi_x)_{y_0} \colon W \to V$ is given by the matrix

$$\left( \sum_j f_{ij}(x) \frac{\partial h_{ij}}{\partial y_k}(y_0) \right)_{(i,k)}$$

whose entries are regular functions on $x$. The claim follows. $\qquad\square$

*Remark* 3.4. In [KR15] we show that the canonical homomorphisms

$$\mathrm{Ad} \colon \mathrm{Aut}(\mathbb{A}^n) \to \mathrm{Aut}_{\mathrm{Lie}}(\mathrm{Lie}\,\mathrm{Aut}(\mathbb{A}^n)) \text{ and } \mathrm{Ad} \colon \mathrm{Aut}(\mathbb{A}^n) \to \mathrm{Aut}_{\mathrm{Lie}}(\mathrm{Lie}\,\mathrm{SAut}(\mathbb{A}^n))$$

are both bijective. This can be improved.

**Proposition 3.5.**
  (1) *The adjoint representation $\mathrm{Ad} \colon \mathrm{Aut}(\mathbb{A}^n) \to \mathrm{Aut}_{\mathrm{Lie}}(\mathrm{Lie}\,\mathrm{Aut}(\mathbb{A}^n))$ is an isomorphism of ind-groups.*
  (2) *The induced map $\rho \colon \mathrm{Aut}_{\mathrm{Lie}}(\mathrm{Lie}\,\mathrm{Aut}(\mathbb{A}^n)) \to \mathrm{Aut}_{\mathrm{Lie}}(\mathrm{Lie}\,\mathrm{SAut}(\mathbb{A}^n))$ is an isomorphism of ind-groups.*

*Proof.* We will use here the identification of $\mathrm{Lie}\,\mathrm{Aut}(\mathbb{A}^n)$ with $\mathrm{Vec}^c(\mathbb{A}^n)$, see Remark 2.6. Put $\partial_{x_i} := \frac{\partial}{\partial x_i}$.

(1) Let $\mathbf{f} = (f_1, \ldots, f_n) \in \operatorname{Aut}(\mathbb{A}^n)$ and set $\theta := \operatorname{Ad}(\mathbf{f}^{-1}) \in \operatorname{Aut}_{\operatorname{Lie}}(\operatorname{Vec}^c(\mathbb{A}^n))$. Then the matrix $\left(\theta(\partial_{x_k})x_j\right)_{(j,k)}$ is invertible, and

$$(*) \qquad \left(\theta(\partial_{x_k})x_j\right)_{(j,k)}^{-1} = \operatorname{Jac}(\mathbf{f}) = \left(\frac{\partial f_j}{\partial x_i}\right)_{(i,j)},$$

see [KR15, Remark 4.2]. We now claim that the map

$$\theta \mapsto \left(\theta(\partial_{x_k})x_j\right)_{(j,k)}^{-1} : \operatorname{Aut}_{\operatorname{Lie}}(\operatorname{Vec}^c(\mathbb{A}^n)) \to \operatorname{M}_n(\mathbb{k}[x_1, \ldots, x_n])$$

is a well-defined morphism of ind-varieties. In fact, $\theta \mapsto \theta(\partial_{x_k})x_j$ is the composition of the orbit map $\theta \mapsto \theta(\partial_{x_k}) \colon \operatorname{Aut}_{\operatorname{Lie}}(\operatorname{Vec}^c(\mathbb{A}^n)) \to \operatorname{Vec}^c(\mathbb{A}^n)$ and the evaluation map $\delta \mapsto \delta(x_j) \colon \operatorname{Vec}^c(\mathbb{A}^n) \to \mathbb{k}[x_1, \ldots, x_n]$, hence $\theta \mapsto \Theta := \left(\theta(\partial_{x_k})x_j\right)_{(j,k)}$ is a morphism. Since $\operatorname{jac}(\Theta) \in \mathbb{k}^*$ the claim follows.

Now recall that the gradient $\mathbb{k}[x_1, \ldots, x_n] \to \mathbb{k}[x_1, \ldots, x_n]^n$, $f \mapsto (\frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n})$, defines an isomorphism

$$\gamma \colon \mathbb{k}[x_1, \ldots, x_n]_{\geq 1} \xrightarrow{\sim} \Gamma := \{(h_1, \ldots, h_n) \mid \frac{\partial h_i}{\partial x_j} = \frac{\partial h_j}{\partial x_i} \text{ for all } i < j\}.$$

It follows from $(*)$ that the rows of the matrix $(h_{ij})_{(i,j)} := \left(\theta(\partial_{x_k})x_j\right)_{(j,k)}^{-1}$ belong to $\Gamma$, so that we get a morphism

$$\psi \colon \operatorname{Aut}_{\operatorname{Lie}}(\operatorname{Vec}^c(\mathbb{A}^n)) \to \mathbb{k}[x_1, \ldots, x_n]^n, \ \theta \mapsto (f_1, \ldots, f_n),$$

where $f_i := \gamma^{-1}(h_{i1}, \ldots, h_{in}) \in \mathbb{k}[x_1, \ldots, x_n]_{\geq 1}$. By construction, we have

$$(**) \qquad \psi(\theta) = \psi(\operatorname{Ad}(\mathbf{f}^{-1})) = \mathbf{f}_0 := (f_1 - f_1(0), \ldots, f_n - f_n(0)) = \mathbf{t}_{-\mathbf{f}(0)} \circ \mathbf{f}$$

where $\mathbf{t}_a$ is the translation $v \mapsto v + a$. Let $S \subseteq \operatorname{Aff}_n$ be the subgroup of translations, and set $\tilde{S} := \operatorname{Ad}(S)$. Then $\tilde{S} \subseteq \operatorname{Aut}_{\operatorname{Lie}}(\operatorname{Vec}^c(\mathbb{A}^n))$ is a closed algebraic subgroup and $\operatorname{Ad} \colon S \to \tilde{S}$ is an isomorphism. It follows from $(**)$ that $\operatorname{Ad}(\psi(\theta)) \cdot \theta = \operatorname{Ad}(\mathbf{t}_{-\mathbf{f}(0)}) \in \tilde{S}$, and so

$$\tilde{\psi}(\theta) := \psi(\theta)^{-1} \cdot (\operatorname{Ad}|_S)^{-1}(\operatorname{Ad}(\psi(\theta)) \cdot \theta)$$

is a well-defined morphism $\tilde{\psi} \colon \operatorname{Aut}_{\operatorname{Lie}}(\operatorname{Vec}^c(\mathbb{A}^n)) \to \operatorname{Aut}(\mathbb{A}^n)$ with the property that

$$\operatorname{Ad}(\tilde{\psi}(\theta)) = \operatorname{Ad}(\psi(\theta)^{-1}) \cdot \operatorname{Ad}(\psi(\theta)) \cdot \theta = \theta.$$

Thus $\operatorname{Ad} \colon \operatorname{Aut}(\mathbb{A}^n) \to \operatorname{Aut}_{\operatorname{Lie}}(\operatorname{Lie}\operatorname{Aut}(\mathbb{A}^n))$ is an isomorphism, with inverse $\tilde{\psi}$.

(2) Clearly, the restriction $\rho \colon \operatorname{Aut}_{\operatorname{Lie}}(\operatorname{Lie}\operatorname{Aut}(\mathbb{A}^n)) \to \operatorname{Aut}_{\operatorname{Lie}}(\operatorname{Lie}\operatorname{SAut}(\mathbb{A}^n))$ is a homomorphism of ind-groups, and it is bijective (Remark 3.4). It follows from (1) that the composition $\rho \circ \operatorname{Ad} \colon \operatorname{Aut}(\mathbb{A}^n) \to \operatorname{Aut}_{\operatorname{Lie}}(\operatorname{Lie}\operatorname{SAut}(\mathbb{A}^n))$ is a bijective homomorphism of ind-groups. Now we use Theorem 1.4(1) to conclude that $\rho \circ \operatorname{Ad}$ is an isomorphism, hence $\rho$ is an isomorphism, too. Note that in the proof of Theorem 1.4(1) below we will only use Proposition 3.5(1). $\qquad \square$

*Proof of Theorem 1.4.* (1) Let $\varphi \colon \operatorname{Aut}(\mathbb{A}^n) \to \mathcal{G}$ be an homomorphism of ind-groups such that $d\varphi$ is injective. We can assume that $\mathcal{G} = \overline{\varphi(\operatorname{Aut}(\mathbb{A}^n))}$, and we will show that $\varphi$ is an isomorphism. The basic idea is to construct a homomorphism $\psi \colon \mathcal{G} \to \operatorname{Aut}(\mathbb{A}^n)$ such that $\psi \circ \varphi = \operatorname{id}$. By Proposition 3.6 below this implies that $\varphi$ is a closed immersion, hence an isomorphism.

Denote by $L \subseteq \operatorname{Lie}\mathcal{G}$ the image of $d\varphi$. For any $g \in \operatorname{Aut}(\mathbb{A}^n)$ we have $d\varphi \circ \operatorname{Ad}(g) = \operatorname{Ad}(\varphi(g)) \circ d\varphi$. In particular, $L$ is stable under $\operatorname{Ad}(\varphi(g))$, hence stable under $\operatorname{Ad}(\mathcal{G})$,

because $\varphi(\mathrm{Aut}(\mathbb{A}^n))$ is dense in $\mathcal{G}$. Thus we get the following commutative diagram of homomorphisms of ind-groups

$$
\begin{array}{ccc}
\mathrm{Aut}(\mathbb{A}^n) & \xrightarrow{\ \ \varphi\ \ } & \mathcal{G} \\
{\scriptstyle \mathrm{Ad}_{\mathrm{Aut}(\mathbb{A}^n)}}\downarrow{\scriptstyle \simeq} & & \downarrow{\scriptstyle \mathrm{Ad}_{\mathcal{G}}} \\
\mathrm{Aut}_{\mathrm{Lie}}(\mathrm{Lie}\,\mathrm{Aut}(\mathbb{A}^n)) & \xrightarrow{\ \simeq\ } & \mathrm{Aut}_{\mathrm{Lie}}(L)
\end{array}
$$

where the first vertical map is an isomorphism by Proposition 3.5(1). Thus, the composition $\mathrm{Ad}_{\mathcal{G}}\circ\varphi\colon \mathrm{Aut}(\mathbb{A}^n) \to \mathrm{Aut}_{\mathrm{Lie}}(L) \simeq \mathrm{Aut}(\mathbb{A}^n)$ is an isomorphism, and so $\varphi$ is also an isomorphism, by Proposition 3.6 below.

If $d\varphi$ is not injective, then $\ker d\varphi \supseteq \mathrm{Lie}\,\mathrm{SAut}(\mathbb{A}^n)$ (Remark 2.3) and so $d\varphi = f \circ d\,\mathrm{jac}$ where $f\colon \Bbbk \to \mathrm{Lie}\,\mathcal{G}$ is a Lie algebra homomorphism. If $\Bbbk^* \subseteq \mathrm{GL}_n(\Bbbk)$ denotes the center, then $\varphi|_{\Bbbk^*}\colon \Bbbk^* \to \mathcal{G}$ factor through $?^n\colon \Bbbk^* \to \Bbbk^*$, because $\mathrm{SL}_n(\Bbbk) \subseteq \ker\varphi$, i.e. $\varphi(z) = \rho(z^n)$ for any $z \in \Bbbk^*$ and a suitable homomorphism $\rho\colon \Bbbk^* \to \mathcal{G}$ of ind-groups. By construction, $d\rho_e = f\colon \Bbbk \to \mathrm{Lie}\,\mathcal{G}$, and so the two homomorphisms $\varphi$ and $\rho \circ \mathrm{jac}$ have the same differential. Thus, by Proposition 2.7, we get $\varphi = \rho \circ \mathrm{jac}$, and we are done.

(2) Let $\varphi\colon \mathrm{SAut}(\mathbb{A}^n) \to \mathcal{G}$ be a homomorphism of ind-groups. If $d\varphi_e$ is not injective, then $d\varphi_e$ is the trivial map (Remark 2.3), hence $d\varphi_e = d\bar\varphi_e$ where $\bar\varphi\colon \mathbf{g} \mapsto e$ is the constant homomorphism. Again by Proposition 2.7 we get $\varphi = \bar\varphi$.

If $d\varphi_e$ is injective, set $L := d\varphi_e(\mathrm{Lie}\,\mathrm{SAut}) \subseteq \mathrm{Lie}\,\mathcal{G}$. Again we can assume that $\mathcal{G} = \overline{\varphi(\mathrm{SAut}(\mathbb{A}^n))}$. Since $L$ is stable under $\mathrm{Ad}\,\varphi(\mathbf{g})$ for all $\mathbf{g} \in \mathrm{Aut}(\mathbb{A}^n)$ it is also stable under $\mathcal{G}$, and we get, as above, the following commutative diagram

$$
\begin{array}{ccccc}
\mathrm{Aut}(\mathbb{A}^n) & \xleftarrow{\ \supseteq\ } & \mathrm{SAut}(\mathbb{A}^n) & \xrightarrow{\ \ \varphi\ \ } & \mathcal{G} \\
{\scriptstyle \mathrm{Ad}_{\mathrm{Aut}(\mathbb{A}^n)}}\downarrow{\scriptstyle \simeq} & & {\scriptstyle \mathrm{Ad}_{\mathrm{SAut}(\mathbb{A}^n)}}\downarrow{\scriptstyle \subseteq} & & \downarrow{\scriptstyle \mathrm{Ad}_{\mathcal{G}}} \\
\mathrm{Aut}_{\mathrm{Lie}}(\mathrm{Lie}\,\mathrm{Aut}(\mathbb{A}^n)) & \xrightarrow[\mathrm{bij}]{\ \Psi\ } & \mathrm{Aut}_{\mathrm{Lie}}(\mathrm{Lie}\,\mathrm{SAut}(\mathbb{A}^n)) & \xrightarrow[\simeq]{\ \Phi\ } & \mathrm{Aut}_{\mathrm{Lie}}(L)
\end{array}
$$

where $\mathrm{Ad}_{\mathrm{Aut}(\mathbb{A}^n)}$ is an isomorphism and $\Psi$ is a bijective homomorphism (see Proposition 3.5 and Remark 3.4). The image $\mathfrak{A} \subseteq \mathrm{Aut}_{\mathrm{Lie}}(\mathrm{Lie}\,\mathrm{SAut}(\mathbb{A}^n))$ of $\mathrm{SAut}(\mathbb{A}^n)$ is a closed subgroup isomorphic to $\mathrm{SAut}(\mathbb{A}^n)$, and $\mathfrak{A} \xrightarrow{\sim} \Phi(\mathfrak{A}) = \mathrm{Ad}_{\mathcal{G}}(\varphi(\mathrm{SAut}(\mathbb{A}^n)))$. But $\varphi(\mathrm{SAut}(\mathbb{A}^n)) \subseteq \mathcal{G}$ is dense, and so $\mathrm{Ad}_{\mathcal{G}}(\mathcal{G}) = \Phi(\mathfrak{A})$. Thus, the composition $\mathrm{Ad}_{\mathcal{G}}\circ\varphi\colon \mathrm{SAut}(\mathbb{A}^n) \to \Phi(\mathfrak{A})$ is an isomorphism, and so $\varphi$ is an isomorphism, by Proposition 3.6 below. $\qquad\square$

**Proposition 3.6.** *Let $\mathcal{H}, \mathcal{G}$ be two ind-groups, and let $\varphi\colon \mathcal{H} \to \mathcal{G}$, $\psi\colon \mathcal{G} \to \mathcal{H}$ be two homomorphisms. If $\psi \circ \varphi = \mathrm{id}_{\mathcal{H}}$, then $\varphi$ is a closed immersion, i.e. $\varphi(\mathcal{H}) \subseteq \mathcal{G}$ is a closed subgroup and $\varphi$ induces an isomorphism $\mathcal{H} \xrightarrow{\sim} \varphi(\mathcal{H})$.*

*Proof.* By base change we can assume that the base field $\Bbbk$ is uncountable. Let $\mathcal{H} = \bigcup_i \mathcal{H}_i$ and $\mathcal{G} = \bigcup_j \mathcal{G}_j$ where we can assume that $\mathcal{H}_i \subseteq \mathcal{G}_i$ for all $i$. Moreover, for every $i$ there is a $k = k(i)$ such that $\psi(\mathcal{G}_i) \subseteq \mathcal{H}_k$. By assumption, the composition $\psi \circ \varphi\colon \mathcal{H}_i \to \mathcal{G}_i \to \mathcal{H}_k$ is the closed embedding $\mathcal{H}_i \hookrightarrow \mathcal{H}_k$, hence the first map is a closed embedding. Thus $H_i := \varphi(\mathcal{H}_i)$ is a closed subset of $\mathcal{G}_i$ and $H := \varphi(\mathcal{H}) = \bigcup_i H_i$. Now the claim follows from Lemma 3.7 below by setting $S := \ker\psi$. $\qquad\square$

Recall that a subset $S \subseteq \mathfrak{V}$ of an ind-variety $\mathfrak{V}$ is called *ind-constructible* if $S = \bigcup_i S_i$ where $S_i \subseteq S_{i+1}$ are constructible subsets of $\mathfrak{V}$.

**Lemma 3.7.** *Let $\mathcal{G}$ be an ind-group, $H \subseteq \mathcal{G}$ a subgroup and $S \subseteq \mathcal{G}$ an ind-constructible subset. Assume that $\Bbbk$ is uncountable and that*

(1) $H = \bigcup_i H_i$ *where* $H_i \subseteq H_{i+1} \subseteq \mathcal{G}$ *are closed algebraic subsets,*

(2) *the multiplication map* $S \times H \to \mathcal{G}$ *is bijective.*

*Then $H$ is a closed subgroup of $\mathcal{G}$.*

*Proof.* Let $\mathcal{G} = \bigcup_k \mathcal{G}_k$. We have to show that for every $k$ there exists an $i = i(k)$ such that $H \cap \mathcal{G}_k = H_i \cap \mathcal{G}_k$. We can assume that $e \in S = \bigcup_i S_i$. Then, by assumption, $\mathcal{G} = \bigcup_j S_j H_j$. Since $S_j H_j \cap \mathcal{G}_k$ is a constructible subset of $\mathcal{G}_k$ it follows that there exists a $j = j(k)$ such that $\mathcal{G}_k \subseteq S_j H_j$ ([FK14, Lemma 3.6.4]). Setting $\dot{S} := S \setminus \{e\}$ we get $\dot{S}H \cap H = \emptyset$. Thus, $\mathcal{G}_k = (\dot{S}_i H_i \cap \mathcal{G}_k) \cup (H_i \cap \mathcal{G}_k)$ and $H \cap \dot{S}_i H_i = \emptyset$, hence $H \cap \mathcal{G}_k = H_i \cap \mathcal{G}_k$.                                                                      $\square$

Finally, we can prove Theorem 1.5.

*Proof of Theorem 1.5.* (1) We already know from Theorem 1.4 that an injective homomorphism $\varphi \colon \mathrm{Aut}(\mathbb{A}^n) \to \mathrm{Aut}(\mathbb{A}^n)$ is a closed immersion. We claim that $d\varphi_e \colon \mathrm{Lie}\,\mathrm{Aut}(\mathbb{A}^n) \to \mathrm{Lie}\,\mathrm{Aut}(\mathbb{A}^n)$ is an isomorphism. To show this, consider the linear action of $\mathrm{GL}_n(\Bbbk)$ on $\mathrm{Lie}\,\mathrm{Aut}(\mathbb{A}^n)$. We then have

$$\mathrm{Lie}\,\mathrm{Aut}(\mathbb{A}^n) \subseteq \mathrm{Vec}(\mathbb{A}^n) \simeq \Bbbk^n \otimes \Bbbk[x_1, \ldots, x_n] = \bigoplus_d \Bbbk^n \otimes \Bbbk[x_1, \ldots, x_n]_d$$

and the latter is multiplicity-free as a $\mathrm{GL}_n(\Bbbk)$-module as well as an $\mathrm{SL}_n(\Bbbk)$-module.

Now $\varphi(\mathrm{GL}_n(\Bbbk)) \subseteq \mathrm{Aut}(\mathbb{A}^n)$ is a closed subgroup isomorphic to $\mathrm{GL}_n(\Bbbk)$. Moreover, $d\varphi_e \colon \mathrm{Lie}\,\mathrm{Aut}(\mathbb{A}^n) \to \mathrm{Lie}\,\mathrm{Aut}(\mathbb{A}^n)$ is an injective linear map which is equivariant with respect to $\varphi \colon \mathrm{GL}_n(\Bbbk) \xrightarrow{\sim} \varphi(\mathrm{GL}_n(\Bbbk))$. Since $\varphi(\mathrm{GL}_n(\Bbbk))$ is conjugate to the standard $\mathrm{GL}_n(\Bbbk) \subseteq \mathrm{Aut}(\mathbb{A}^n)$ and since the representation of $\mathrm{GL}_n(\Bbbk)$ on $\mathrm{Lie}\,\mathrm{Aut}(\mathbb{A}^n)$ is multiplicity-free, it follows that $d\varphi_e$ is an isomorphism. Thus $\mathcal{G} := \varphi(\mathrm{Aut}(\mathbb{A}^n)) \subseteq \mathrm{Aut}(\mathbb{A}^n)$ is a closed subgroup with the same Lie algebra as $\mathrm{Aut}(\mathbb{A}^n)$, and we get the following commutative diagram (see proof of Theorem 1.4):

$$
\begin{array}{ccccc}
\mathrm{Aut}(\mathbb{A}^n) & \xrightarrow{\ \varphi\ } & \mathcal{G} & \xrightarrow{\ \subseteq\ } & \mathrm{Aut}(\mathbb{A}^n) \\
{\scriptstyle \mathrm{Ad}_{\mathrm{Aut}(\mathbb{A}^n)}}\Big\downarrow {\scriptstyle \simeq} & & \Big\downarrow {\scriptstyle \mathrm{Ad}_{\mathcal{G}}} & & \Big\downarrow {\scriptstyle \mathrm{Ad}_{\mathrm{Aut}(\mathbb{A}^n)}} \\
\mathrm{Aut}_{\mathrm{Lie}}(\mathrm{Lie}\,\mathrm{Aut}(\mathbb{A}^n))) & \xrightarrow{\ \simeq\ } & \mathrm{Aut}_{\mathrm{Lie}}(\mathrm{Lie}\,\mathcal{G}) & =\!=\!= & \mathrm{Aut}_{\mathrm{Lie}}(\mathrm{Lie}\,\mathrm{Aut}(\mathbb{A}^n))
\end{array}
$$

As a consequence, all maps are isomorphisms, and so $\mathcal{G} = \mathrm{Aut}(\mathbb{A}^n)$ and $\varphi$ is an isomorphism.

It remains to see that every automorphism $\varphi \in \mathrm{Aut}(\mathbb{A}^n)$ is inner. Since $d\varphi_e \in \mathrm{Aut}_{\mathrm{Lie}}(\mathrm{Lie}\,\mathrm{Aut}(\mathbb{A}^n))$ we get $d\varphi_e = \mathrm{Ad}(\mathbf{g})$ for some $\mathbf{g} \in \mathrm{Aut}(\mathbb{A}^n)$, by Remark 3.4. This means that $d\varphi_e = (d\,\mathrm{Int}\,\mathbf{g})_e$ and so $\varphi = \mathrm{Int}\,\mathbf{g}$, by Proposition 2.7.

(2) The same argument as above shows that every nontrivial homomorphism $\mathrm{SAut}(\mathbb{A}^n) \to \mathrm{SAut}(\mathbb{A}^n)$ is an isomorphism where we use the fact that the action of $\mathrm{SL}_n(\Bbbk)$ on $\mathrm{Lie}\,\mathrm{SAut}(\mathbb{A}^n)$ is multiplicity-free.

Moreover, the homomorphism $\mathrm{Ad} \colon \mathrm{Aut}(\mathbb{A}^n) \to \mathrm{Aut}_{\mathrm{Lie}}(\mathrm{Lie}\,\mathrm{SAut}(\mathbb{A}^n))$ is a bijective homomorphism of ind-groups (Remark 3.4). Hence, for every $\varphi \in \mathrm{SAut}(\mathbb{A}^n)$ there is a $\mathbf{g} \in \mathrm{Aut}(\mathbb{A}^n)$ such that $d\varphi_e = \mathrm{Ad}\,\mathbf{g}$ which implies that $\varphi = \mathrm{Int}\,\mathbf{g}$.      $\square$

## 4. A SPECIAL SUBGROUP OF $\mathrm{Aut}(X)$

Our Theorem 1.1 will follow from a more general result which we will describe now. For any affine variety $X$ consider the normal subgroup $\mathcal{U}(X)$ of $\mathrm{Aut}(X)$ generated by the unipotent elements, or, equivalently, by the closed subgroups isomorphic to $\Bbbk^+$. This is an instance of a so-called *algebraically generated subgroup* of an ind-group, see [KZ14]. The group $\mathcal{U}(X)$ was introduced and studied in [AFK13] where the authors called it the *group of special automorphisms of $X$*. In particular, they proved a very interesting connection between transitivity properties of $\mathcal{U}(X)$ and the flexibility of $X$.

Let us define the following notion of an "algebraic" homomorphism between these groups.

**Definition 4.1.** A homomorphism $\varphi\colon \mathcal{U}(X) \to \mathcal{U}(Y)$ is *algebraic*, if for any closed subgroup $U \subseteq \mathcal{U}(X)$ isomorphic to $\Bbbk^+$ the image $\varphi(U) \subseteq \mathcal{U}(Y)$ is closed and $\varphi|_U\colon U \to \varphi(U)$ is a homomorphism of algebraic groups. We say that $\mathcal{U}(X)$ and $\mathcal{U}(Y)$ are *algebraically isomorphic*, $\mathcal{U}(X) \simeq \mathcal{U}(Y)$, if there exists a bijective homomorphism $\varphi\colon \mathcal{U}(X) \to \mathcal{U}(Y)$ such that $\varphi$ and $\varphi^{-1}$ are both algebraic.

**Lemma 4.2.** *Let $\varphi\colon \mathcal{U}(X) \to \mathcal{U}(Y)$ be an algebraic homomorphism. Then, for any algebraic subgroup $G \subseteq \mathcal{U}(X)$ generated by unipotent elements the image $\varphi(G) \subseteq \mathcal{U}(Y)$ is closed and $\varphi|_G\colon G \to \varphi(G)$ is a homomorphism of algebraic groups.*

*Proof.* There exist closed subgroups $U_1, \ldots, U_m \subseteq G$ isomorphic to $\Bbbk^+$ such that the multiplication map $\mu\colon U_1 \times U_2 \times \cdots \times U_m \to G$ is surjective. This gives the following commutative diagram

$$
\begin{array}{ccc}
U_1 \times U_2 \times \cdots \times U_m & \xrightarrow{\;\;\mu\;\;} & G \\[2pt]
\Big\downarrow{\scriptstyle \tilde\varphi := \varphi|_{U_1} \times \cdots \times \varphi|_{U_m}} & & \Big\downarrow{\scriptstyle \varphi|_G} \\[2pt]
\varphi(U_1) \times \varphi(U_2) \times \cdots \times \varphi(U_m) & \xrightarrow{\;\;\bar\mu\;\;} & \varphi(G)
\end{array}
$$

where all maps are surjective. It follows that $\overline{\varphi(G)} \subseteq \mathrm{Aut}(Y)$ is a (closed) algebraic subgroup, and thus $\varphi(G) = \overline{\varphi(G)}$, because $\varphi(G)$ is constructible. It remains to show that $\varphi|_G$ is a morphism. This follows from the next lemma, because $G$ is normal, and $\mu$ and the composition $\varphi|_G \circ \mu = \bar\mu \circ \tilde\varphi$ are both morphisms. $\square$

**Lemma 4.3.** *Let $X, Y, Z$ be irreducible affine varieties where $Y$ is normal. Let $\mu\colon X \to Y$ be a surjective morphism and $\varphi\colon Y \to Z$ an arbitrary map. If the composition $\varphi \circ \mu$ is a morphism, then $\varphi$ is a morphism.*

*Proof.* We have the following commutative diagram of maps

$$
\begin{array}{ccc}
\Gamma_{\varphi\circ\mu} & \xrightarrow{\;\;\subseteq\;\;} & X \times Z \\[2pt]
\Big\downarrow{\scriptstyle \bar\mu} & & \Big\downarrow{\scriptstyle \mu \times \mathrm{id}} \\[2pt]
\Gamma_{\varphi} & \xrightarrow{\;\;\subseteq\;\;} & Y \times Z \\[2pt]
\Big\downarrow{\scriptstyle p} & & {\scriptstyle \mathrm{pr}_Y}\Big\downarrow \\[2pt]
Y & =\!=\!=\!= & Y
\end{array}
$$

where $\Gamma_{\varphi\circ\mu}$ and $\Gamma_{\varphi}$ denote the graphs of the corresponding maps. We have to show that $\Gamma_{\varphi} \subseteq Y \times Z$ is closed and that $p$ is an isomorphism. The diagram shows that $\bar\mu$ is

surjective, hence $\Gamma_\varphi$ is constructible, and $p$ is bijective. Thus, the induced morphism $\bar{p}\colon \overline{\Gamma_\varphi} \to Y$ is birational and surjective, hence an isomorphism since $Y$ is normal (see [Igu73, Lemma 4, page 379]). Since $p$ is bijective, we finally get $\Gamma_\varphi = \overline{\Gamma_\varphi}$.   $\square$

The proof of the following theorem will be given in the next section.

**Theorem 4.4.** *Let $X$ be a connected affine variety. If $\mathcal{U}(X)$ is algebraically isomorphic to $\mathcal{U}(\mathbb{A}^n)$, then $X$ is isomorphic to $\mathbb{A}^n$.*

A first consequence is our Theorem 1.1.

**Corollary 4.5.** *Let $X$ be a connected affine variety. If $\mathrm{Aut}(X) \simeq \mathrm{Aut}(\mathbb{A}^n)$ as ind-groups, then $X \simeq \mathbb{A}^n$.*

*Proof.* It is clear from the definition that an isomorphism $\mathrm{Aut}(X) \xrightarrow{\sim} \mathrm{Aut}(\mathbb{A}^n)$ induces an algebraic isomorphism $\mathcal{U}(X) \xrightarrow{\sim} \mathcal{U}(\mathbb{A}^n)$.   $\square$

Finally, we define the following closed subgroups of $\mathrm{Aut}(X)$:
$$\mathrm{Aut}^{alg}(X) := \overline{\langle G \mid G \subseteq \mathrm{Aut}(X) \text{ connected algebraic}\rangle},$$
$$\mathrm{SAut}^{alg}(X) := \overline{\langle U \mid U \subseteq \mathrm{Aut}(X) \text{ unipotent algebraic}\rangle}.$$
We have $\mathrm{SAut}^{alg}(X) = \overline{\mathcal{U}(X)} \subseteq \mathrm{Aut}^{alg}(X) \subseteq \mathrm{Aut}(X)$. Now the same argument as above gives the next result.

**Corollary 4.6.** *Let $X$ be a connected affine variety. If $\mathrm{SAut}^{alg}(X)$ is isomorphic to $\mathrm{SAut}^{alg}(\mathbb{A}^n)$ as ind-groups, then $X$ is isomorphic to $\mathbb{A}^n$, and the same holds if we replace $\mathrm{SAut}^{alg}$ by $\mathrm{Aut}^{alg}$.*

## 5. Root subgroups and modifications

Let $\mathcal{G}$ be an ind-group, and let $T \subseteq \mathcal{G}$ be a torus.

**Definition 5.1.** A closed subgroup $U \subseteq \mathcal{G}$ isomorphic to $\mathbb{k}^+$ and normalized by $T$ is called a *root subgroup* with respect to $T$. The character of $T$ on $\mathrm{Lie}\, U \simeq \mathbb{k}$ is called the *weight of $U$*.

Let $X$ be an affine variety and consider a nontrivial action of $\mathbb{k}^+$ on $X$, given by $\lambda\colon \mathbb{k}^+ \to \mathrm{Aut}(X)$. If $f \in \mathcal{O}(X)$ is $\mathbb{k}^+$-invariant, then we define the *modification* $f \cdot \lambda$ of $\lambda$ in the following way (see [FK14, section 8.3]):
$$(f \cdot \lambda)(s)x := \lambda(f(x)s)x \quad \text{for} \;\; s \in \mathbb{k} \text{ and } x \in X.$$
It is easy to see that this is again a $\mathbb{k}^+$-action. In fact, if the corresponding locally nilpotent vector field is $\delta_\lambda$, the $f\delta_\lambda$ is again locally nilpotent, because $f$ is $\lambda$-invariant, and defines the modified $\mathbb{k}^+$-action $f \cdot \lambda$. This modified action $f \cdot \lambda$ is trivial if and only if $f$ vanishes on every irreducible component $X_i$ of $X$ where the action $\lambda$ is nontrivial. It is clear that the orbits of $f \cdot \lambda$ are contained in the orbits of $\lambda$ and that they are equal on the open subset $X_f \subseteq X$. In particular, if $X$ is irreducible and $f \neq 0$, then $\lambda$ and $f \cdot \lambda$ have the same invariants.

If $U \subseteq \mathrm{Aut}(X)$ is isomorphic to $\mathbb{k}^+$ and if $f \in \mathcal{O}(X)^U$ is a $U$-invariant, then we define in a similar way the *modification $f \cdot U$* of $U$. Choose an isomorphism $\lambda\colon \mathbb{k}^+ \xrightarrow{\sim} U$ and set $f \cdot U := (f \cdot \lambda)(\mathbb{k}^+)$, the image of the modified action. Note that $\mathrm{Lie}(f \cdot U) = f\,\mathrm{Lie}\, U \subseteq \mathrm{Lie}\,\mathrm{Aut}(X) \subseteq \mathrm{Vec}(X)$ where we use the fact that $\mathrm{Vec}(X)$ is a $\mathcal{O}(X)$-module.

If a torus $T$ acts linearly and rationally on a vector space $V$ of countable dimension, then we call $V$ *multiplicity free* if the weight spaces $V_\alpha$ are all of dimension $\leq 1$. The following lemma is crucial.

**Lemma 5.2.** *Let $X$ be an irreducible affine variety, and let $T \subseteq \mathrm{Aut}(X)$ be a torus. Assume that there exists a root subgroup $U \subseteq \mathrm{Aut}(X)$ with respect to $T$ such that $\mathcal{O}(X)^U$ is multiplicity-free. Then $\dim T \leq \dim X \leq \dim T + 1$.*

*Proof.* The first inequality $\dim T \leq \dim X$ is clear, because $T$ acts faithfully on $X$. It follows from [KD14]) that there exists a $T$-semi-invariant $f \in \mathcal{O}(X)^U$ such that $\mathcal{O}(X)^U_f$ is finitely generated. Clearly, $\mathcal{O}(X)^U_f$ is $T$-stable and multiplicity-free. The algebra $\mathcal{O}(X)^U_f = \mathcal{O}(X_f)^U$ is the coordinate ring of the algebraic quotient $Z := X_f /\!\!/ U$ on which $T$ acts. It follows from [Kra84, II.3.4 Satz 5]) that $T$ has a dense orbit in $Z$, and so $\dim Z \leq \dim T$. Since $\dim Z = \dim X_f /\!\!/ U = \dim X_f - 1 = \dim X - 1$, we get the second inequality. $\square$

**Lemma 5.3.** *We have $\mathcal{U}(\mathbb{A}^n) \subseteq \mathrm{SAut}(\mathbb{A}^n)$, and its closure $\overline{\mathcal{U}(\mathbb{A}^n)}$ is connected. Moreover, $\mathrm{Lie}\,\overline{\mathcal{U}(\mathbb{A}^n)} = \mathrm{Lie}\,\mathrm{SAut}(\mathbb{A}^n)$, hence it is a simple Lie algebra.*

*Proof.* The first statement is obvious, since every unipotent algebraic group is contained in $\mathrm{SAut}(\mathbb{A}^n)$. The second claim follows from $\mathcal{U}(\mathbb{A}^n) \subseteq \overline{\mathcal{U}(\mathbb{A}^n)}^\circ$ (see Lemma 6.3 in the next section). For the last statement we remark that $\mathrm{Lie}\,\mathrm{SAut}(\mathbb{A}^n)$ is generated by the Lie algebras of the algebraic subgroups (Remark 2.3). $\square$

The group $\mathcal{U}(\mathbb{A}^n)$ contains the normal subgroup generated by all tame elements. But we do not know if $\mathcal{U}(\mathbb{A}^n) = \mathrm{SAut}(\mathbb{A}^n)$ or at least $\overline{\mathcal{U}(\mathbb{A}^n)} = \mathrm{SAut}(\mathbb{A}^n)$, except for $n = 2$ where this is well-known.

Denote by $T_n \subseteq \mathrm{GL}_n(\mathbb{k}) \subseteq \mathrm{Aut}(\mathbb{A}^n)$ the diagonal torus and set $T'_n := T_n \cap \mathrm{SL}_n(\mathbb{k})_n$. The next result can be found in [Lie11, Theorem 1].

**Lemma 5.4.** *Root subgroups of $\mathrm{Aut}(\mathbb{A}^n)$ with respect to $T'_n$ exist, and they have different weights.*

*Proof of Theorem 4.4.* Note that $\mathrm{SL}_n(\mathbb{k})$ and $\mathrm{SAff}_n(\mathbb{k})$ both belong to $\mathcal{U}(\mathbb{A}^n)$ as well as all root subgroups $U$. Fix an algebraic isomorphism $\varphi \colon \mathcal{U}(\mathbb{A}^n) \xrightarrow{\sim} \mathcal{U}(X)$ and denote by $T'$ the image of $T'_n$.

(a) Assume first that $X$ is irreducible. By Lemma 5.4, there exists a root subgroup $U \subseteq \mathcal{U}(X)$ with respect to $T'$, and all root subgroups have different weights. In particular, the root subgroups from $\mathcal{O}(X)^U \cdot U \subseteq \mathcal{U}(X)$ have different weights which implies that $\mathcal{O}(X)^U$ is multiplicity-free, because the map $\mathcal{O}(X)^U \to \mathcal{O}(X)^U \cdot U$ is injective. Hence, by Lemma 5.2, $\dim X \leq \dim T' + 1 = n$, and the claim follows from Proposition 2.8.

(b) Let $X = \bigcup_i X_i$ be the decomposition into irreducible components. Since $\overline{\mathcal{U}(X)}$ is connected by Lemma 5.3 it follows that the components $X_i$ are stable under $\overline{\mathcal{U}(X)}$. Moreover, at least one of the restriction morphisms $\rho_i \colon \mathcal{U}(X) \to \mathcal{U}(X_i)$, say $\rho_1$, is injective on the image $\varphi(\mathrm{SAff}_n) \subseteq \mathcal{U}(X)$, because every nontrivial normal closed subgroup of $\mathrm{SAff}_n$ contains the translations. Let $T_1 := \rho_1(T') \subseteq \mathcal{U}(X_1)$ be the image of $T'$. Choose a root subgroup $U \subseteq \mathcal{U}(X)$ in the image $\varphi(\mathrm{SL}_n(\mathbb{k}))$, and denote by $U_1 := \rho_1(U)$ its image in $\mathcal{U}(X_1)$. Then $U$ is a maximal unipotent subgroup of a closed subgroup of $\mathcal{U}(X)$ isomorphic to $\mathrm{SL}_2(\mathbb{k})$ which implies that the restriction morphism $\rho_1^* \colon \mathcal{O}(X)^U \to \mathcal{O}(X_1)^{U_1}$ is surjective. Since the root subgroups of $\mathcal{U}(X)$

have all different weights this also holds for the root subgroups in $\mathcal{O}(X_1)^{U_1} \cdot U_1$. Hence the $T_1$-action on $\mathcal{O}(X_1)^{U_1}$ is multiplicity-free and so $\dim X_1 \leq n$. Thus $X_1 \simeq \mathbb{A}^n$, by Proposition 2.8, which implies that $X = X_1 \simeq \mathbb{A}^n$, because $X$ is connected.                                                                                       $\square$

## 6. Finite dimensional automorphism groups

It is well-known that for a smooth affine curve $C$ the automorphism group $\operatorname{Aut}(C)$ is finite except for $C \simeq \mathbb{k}, \mathbb{k}^*$. We will see in the next section that every finite group appears as automorphism group of a smooth affine curve. There also exist examples of smooth affine surfaces with a discrete non-finite automorphism group, see [FK14, Proposition 7.2.5]. Recall that an ind-group $\mathcal{G} = \bigcup_k \mathcal{G}_k$ is called *discrete* if $\mathcal{G}_k$ is finite for all $k$, or equivalently, if $\operatorname{Lie}\mathcal{G} = \{0\}$.

**Definition 6.1.** An ind-group $\mathcal{G} = \bigcup_k \mathcal{G}_k$ is called *finite dimensional*, $\dim \mathcal{G} < \infty$, if $\dim \mathcal{G}_k$ is bounded above. In this case we put $\dim \mathcal{G} := \max_k \dim \mathcal{G}_k$.

**Definition 6.2.** Let $\mathcal{G} = \bigcup_k \dim \mathcal{G}_k$ be an ind-group. Define

$$\mathcal{G}^\circ := \bigcup_k \mathcal{G}_k^\circ$$

where $\mathcal{G}_k^\circ$ denotes the connected component of $\mathcal{G}_k$ which contains $e \in \mathcal{G}$.

Recall that an ind-group is called *connected* if for every $g \in \mathcal{G}$ there is an irreducible curve $D$ and a morphism $D \to \mathcal{G}$ whose image contains $e$ and $g$. This implies that for every $k$ there is a $k' \geq k$ and an irreducible component $C \subseteq \mathcal{G}_{k'}$ which contains $\mathcal{G}_k$ (cf. [FK14, Lemma 8.1.2]).

**Lemma 6.3.** *Let $\mathcal{G} = \bigcup_k \mathcal{G}_k$ be an ind-group.*

(1) *$\mathcal{G}^\circ \subseteq \mathcal{G}$ is a connected closed normal subgroup of countable index. In particular, $\operatorname{Lie}\mathcal{G} = \operatorname{Lie}\mathcal{G}^\circ$.*
(2) *We have $\dim \mathcal{G} < \infty$ if and only if $\mathcal{G}^\circ \subseteq \mathcal{G}$ is an algebraic group.*
(3) *We have $\dim \mathcal{G} < \infty$ if and only if $\dim \operatorname{Lie}\mathcal{G} < \infty$.*

*Proof.* (1) For $g \in \mathcal{G}$ and any $k$ the subset $g\mathcal{G}_k^\circ g^{-1}$ is closed, connected, contains $e$, and is contained in some $\mathcal{G}_{k'}$. Hence $g\,\mathcal{G}_k^\circ\,g^{-1} \subseteq \mathcal{G}_{k'}^\circ \subseteq \mathcal{G}^\circ$ which shows that $\mathcal{G}^\circ$ is normal. If $\mathcal{G}^\circ$ meets a connected component $C$ of some $\mathcal{G}_k$, then there is a $k' \geq k$ such that $\mathcal{G}_{k'}^\circ \cap C \neq \emptyset$, hence $C \subseteq \mathcal{G}_{k'}^\circ$. Thus $\mathcal{G}^\circ \cap \mathcal{G}_k = \mathcal{G}_{k''}^\circ \cap \mathcal{G}_k$ for large $k'' \geq k$ and so $\mathcal{G}^\circ$ is closed.

To see that $\mathcal{G}^\circ$ is connected it suffices to show the following. If $C_0, C_1 \subseteq \mathcal{G}_k^\circ$ are irreducible components with $C_0 \cap C_1 \neq \emptyset$, then there is a $k' \geq k$ and an irreducible component $C$ of $\mathcal{G}_{k'}^\circ$ which contains $C_0 \cup C_1$. For this, we choose an $h \in C_0 \cap C_1$ and define the morphism $\mu \colon C_0 \times C_1 \to \mathcal{G}$ by $(g_0, g_1) \mapsto g_0 h^{-1} g_1$. Then the image $Y$ of $\mu$ contains $C_0$ and $C_1$, and the closure $\bar{Y}$ is irreducible. Hence, $\bar{Y}$ is contained in an irreducible component $C$ of $\mathcal{G}_{k'}$, and $C \subseteq \mathcal{G}_{k'}^\circ$, because $C_0, C_1 \subseteq \mathcal{G}_k^\circ \subseteq \mathcal{G}_{k'}^\circ$.

For the last claim we choose elements $g_1, \ldots, g_m \in \mathcal{G}_k$, one from each connected component. Then $\mathcal{G}_k \subseteq g_1 \mathcal{G}^\circ \cup g_2 \mathcal{G}^\circ \cup \cdots \cup g_m \mathcal{G}^\circ$ which implies that $\mathcal{G}/\mathcal{G}^\circ$ is countable.

(2) Assume that $\dim \mathcal{G} < \infty$. Then there is a $k_0$ such that $\dim \mathcal{G}_k^\circ = \dim \mathcal{G}_{k_0}^\circ$ for all $k \geq k_0$. Since $\mathcal{G}_k^\circ$ is contained in an irreducible component $C$ of $\mathcal{G}_{k'}^\circ$ for some $k' > k$, it follows that $\mathcal{G}_k^\circ = C$ in case $k \geq k_0$. Thus $\bigcup_k \mathcal{G}_k^\circ = \mathcal{G}_{k_0}^\circ$, and this is a closed

irreducible algebraic subset, hence an algebraic group. The reverse implication is clear.

(3) If $\dim \mathcal{G} < \infty$, then $\mathcal{G}^\circ$ is an algebraic group, by (2). Since $\operatorname{Lie} \mathcal{G} = \operatorname{Lie} \mathcal{G}^\circ$ by (1) we see that $\operatorname{Lie} \mathcal{G}$ is finite dimensional. If $\dim \operatorname{Lie} \mathcal{G} < \infty$, then $\dim T_g \mathcal{G} < \infty$, because $T_g \mathcal{G}_k \simeq T_e g^{-1} \mathcal{G}_k \subseteq \operatorname{Lie} \mathcal{G}$. Thus $\dim \mathcal{G}_k$ is bounded by $\dim \operatorname{Lie} \mathcal{G}$. $\square$

**Example 6.4.** (1) We have $\operatorname{Aut}(\Bbbk^*) \simeq \mathbb{Z}/2 \ltimes \Bbbk^*$, hence $\operatorname{Aut}(\Bbbk^*)^\circ \simeq \Bbbk^*$. Similarly, $\operatorname{Aut}(\Bbbk^{*n}) \simeq \operatorname{GL}_n(\mathbb{Z}) \ltimes \Bbbk^{*n}$, and so $\operatorname{Aut}(\Bbbk^{*n})^\circ \simeq \Bbbk^{*n}$.

(2) Let $C$ be a smooth curve with trivial automorphism group, and consider the one dimensional variety $Y_C = \mathbb{A}^1 \cup C$ where the two irreducible components meet in $\{0\} \in \mathbb{A}^1$. Then $\operatorname{Aut}(Y_C) \simeq \Bbbk^*$. Moreover, the disjoint union $Y_{C_1} \cup Y_{C_2} \cup \cdots \cup Y_{C_m}$ with pairwise non-isomorphic curves $C_i$ has automorphism group $\Bbbk^{*m}$. We do not know if there is an irreducible variety whose automorphism group is a given torus.

The proof of Theorem 1.3 follows immediately from the next result.

**Proposition 6.5.** *Let $X$ be a connected affine variety. If $X$ admits a nontrivial action of the additive group $\Bbbk^+$, then either $X \simeq \mathbb{A}^1$ or $\dim \operatorname{Aut}(X) = \infty$.*

*Proof.* If $X$ contains a one-dimensional irreducible component $X_i$ with a nontrivial action of $\Bbbk^+$, then $X_i$ is an orbit under $\Bbbk^+$, hence $X = X_i \simeq \mathbb{A}^1$. Otherwise, $\Bbbk^+$ acts non-trivially on an irreducible component $X_j$ of dimension $\geq 2$. Denote by $U \subseteq \operatorname{Aut}(X)$ the image of $\Bbbk^+$. We claim that the unipotent subgroup $\mathcal{O}(X)^U \cdot U \subseteq \operatorname{Aut}(X)$ is infinite dimensional. This follows if we show that the image of $\mathcal{O}(X)^U$ in $\mathcal{O}(X_j)$ is infinite dimensional. For that we first remark that there is a nonzero $U$-invariant $f$ which vanishes on all $X_j \cap X_k$ for $k \neq j$, because the vanishing ideal is $U$-stable. This implies that $X_f \subseteq X_j$, and so

$$\mathcal{O}(X)_f^U = \mathcal{O}(X_f)^U = \mathcal{O}(X_j)_f^U = (\mathcal{O}(X)^U|_{X_j})_f.$$

Thus the image $\mathcal{O}(X)^U|_{X_j} \subseteq \mathcal{O}(X_j)$ is infinite dimensional. $\square$

## 7. Automorphism groups of curves

In this last section we prove Theorem 1.2 which claims that every finite group appear as the automorphism group of a smooth affine curve.

**Lemma 7.1.** *Let $C$ be a smooth affine curve which is neither rational nor elliptic, and let $G \subseteq \operatorname{Aut}(C)$ be a subgroup. Then there is an open subset $C' \subseteq C$ such that $\operatorname{Aut}(C') = G$.*

*Proof.* The curve $C$ is contained as an open set in a smooth projective curve $\bar{C}$. Since every automorphism $C$ extends to $\bar{C}$ we have $G \subseteq \operatorname{Aut}(\bar{C})$. By assumption $\bar{C}$ has genus $> 1$ and so $\operatorname{Aut}(\bar{C})$ is finite. It follows that there is a point $c \in C$ with trivial stabilizer in $\operatorname{Aut}(\bar{C})$. Removing the $G$-orbit $G \cdot c$ we get $\operatorname{Aut}(C \setminus G \cdot c) = G$. In fact, $G \subseteq \operatorname{Aut}(C \setminus G \cdot c)$, and we have equality, because every automorphism of $C \setminus G \cdot c$ extends to $\bar{C}$. $\square$

The lemma shows that we can prove Theorem 1.2 by constructing, for every large $n$, a smooth affine curve $C_n$ such that the symmetric group $\mathcal{S}_n$ appears as a

subgroup of $\mathrm{Aut}(C_n)$. The following construction was suggested by JEAN-PHILIPPE FURTER. We start with the standard action of $\mathcal{S}_n$ on $\Bbbk^n$ and consider the morphism

$$\varphi := (s_1, s_2, \ldots, s_{n-1})\colon \Bbbk^n \to \Bbbk^{n-1}$$

where $s_j \in \Bbbk[x_1, \ldots, x_n]^{\mathcal{S}_n}$ is the $j$th elementary symmetric function. The sequence $s_1, s_2, \ldots, s_n$ is a homogeneous system of parameters, and so the fibers of the morphism $\varphi$ are complete intersections of dimension 1 which are stable under the action of $\mathcal{S}_n$. Note that for the hyperplane $H := \mathcal{V}(x_n)$ the induced morphism $\varphi_H := \varphi|_H\colon H \to \Bbbk^{n-1}$ is the quotient morphism under the action of $\mathcal{S}_{n-1}$.

**Proposition 7.2.** *The general fiber of $\varphi$ is irreducible, i.e. there is a dense open set $U \subseteq \Bbbk^{n-1}$ such that $\varphi^{-1}(u)$ is an irreducible curve for all $u \in U$.*

*Proof.* We will give the proof for $n \geq 5$ which is sufficient for our application. By [Gro66, Proposition 9.7.8] we have to show that the generic fiber is geometrically irreducible, or equivalently that $\Bbbk(s_1, \ldots, s_{n-1})$ is algebraically closed in $\Bbbk(x_1, \ldots, x_n)$. Consider the integral closure $A$ of $\Bbbk[s_1, \ldots, s_{n-1}]$ in $\Bbbk[x_1, \ldots, x_n]$, and denote by $\eta\colon Y \to \Bbbk^{n-1}$ the corresponding finite morphism. Clearly, we have a factorization $\varphi\colon \Bbbk^n \xrightarrow{\bar\varphi} Y \xrightarrow{\eta} \Bbbk^{n-1}$. Moreover, $A$ is stable under the action of $\mathcal{S}_n$, $A^{\mathcal{S}_n} = \Bbbk[s_1, \ldots, s_{n-1}]$, and thus $\eta$ is the quotient under the action of $\mathcal{S}_n$ on $Y$. Restricting $\bar\varphi$ to $H$ we get a similar factorization $\varphi_H\colon H \xrightarrow{\bar\varphi_H} Y \xrightarrow{\eta} \Bbbk^{n-1}$ with a finite $\mathcal{S}_{n-1}$-equivariant morphism $\bar\varphi_H$.



It follows that $A^{\mathcal{S}_{n-1}} = \Bbbk[s_1, \ldots, s_{n-1}] = A^{\mathcal{S}_n}$ which implies that the action of $\mathcal{S}_n$ on $Y$ cannot be faithful. Assuming $n \geq 5$ we deduce that the alternating group $\mathfrak{A}_n$ acts trivially on $Y$.

If $\mathcal{S}_n$ acts trivially, then $Y \xrightarrow{\sim} \Bbbk^{n-1}$ and we are done. Otherwise, $Y = \Bbbk^{n-1}/\mathcal{A}_{n-1}$ and we get a decomposition $A = \Bbbk[s_1, \ldots, s_{n-1}] \oplus \Bbbk[s_1, \ldots, s_{n-1}]f$ where $f$ is a $\mathcal{S}_{n-1}$-semi-invariant and $f^2 \in \Bbbk[s_1, \ldots, s_{n-1}]$.

But $A$ is stable under $\mathcal{S}_n$, and so $f$ is also an $\mathcal{S}_n$-semi-invariant, i.e. $f = d \cdot h$ where $d = \prod_{i<j}(x_i - x_j)$ and $h \in \Bbbk[x_1, \ldots, x_n]^{\mathcal{S}_n}$. Clearly, $f$ vanishes on the hyperplanes $H_{ij} := \mathcal{V}(x_i - x_j)$. On the other hand, we claim that the images $\varphi(H_{ij})$ are dense in $\Bbbk^{n-1}$. Hence $f^2 = 0$ on $\Bbbk^{n-1}$, and so $f = 0$, a contradiction.

In order to prove the claim we remark that $\Bbbk[s_1, \ldots, s_{n-1}] = \Bbbk[p_1, \ldots, p_{n-1}]$ where $p_j := x_1^j + \cdots + x_n^j$. Using these functions to define $\varphi\colon \Bbbk^n \to \Bbbk^{n-1}$ we see that the Jacobian matrix is given by

$$\mathrm{Jac}(\varphi) = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ x_1 & x_2 & \cdots & x_n \\ x_1^2 & x_2^2 & \cdots & x_n^2 \\ \vdots & \vdots & & \vdots \\ x_1^{n-2} & x_2^{n-2} & \cdots & x_n^{n-2} \end{bmatrix}.$$

Every $(n-1) \times (n-1)$-minor is a VANDERMONDE determinant, and so $\mathrm{Jac}(\varphi)$ has rank $n-1$ in all points $a = (a_1, \ldots, a_n)$ where at most two coordinates are equal. Hence $\mathrm{Jac}(\varphi)$ has maximal rank on a dense open set of every hyperplane $H_{ij}$, and so $\varphi|_{H_{ij}}$ is dominant. $\qquad\square$

*Proof of Theorem 1.2.* We can embed every finite group $G$ into some $\mathcal{S}_n$ where we can assume that $n \geq 5$. By Proposition 7.2 the general fiber of $\varphi\colon \Bbbk^n \to \Bbbk^{n-1}$ is an irreducible curve $C$ with a faithful action of $\mathcal{S}_n$. This action lifts to a faithful action on the normalization $\tilde{C}$. Since $n \geq 5$, $\tilde{C}$ is neither rational nor elliptic, hence the claim follows from Lemma 7.1. $\qquad\square$

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